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Bifurcation of the self-excited oscillations of a plate with slight damping in a supersonic gas flow☆

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ABSTRACT

A non-linear boundary-value problem is considered which simulates the oscillations of a plate in a supersonic gas flow. The classical version of the formulation of the problem, proposed by Bolotin, as well as several of its modifications considered by Holmes and Marsden, are taken as a basis. The oscillations of the plate are studied assuming that the damping coefficient is small. This version of the formulation of the problem leads to the need to investigate the bifurcations of the self-excited oscillations in a non-linear boundary-value problem in a case which is close to the critical case of a double pair of pure imaginary values of the stability spectrum. The bifurcation problem is reduced to the investigation of a complex second order non-linear differential equation by the method of normal forms. All the stages in the investigation are carried out without using the Bubnov method.

1. Introduction. Formulation of the problem

One of the best known problems of the oscillations of rigid bodies in a gas or liquid flow is the investigation of panel flutter in the linear and non-linear formulations (see Ref. 1 and 2). As previously, we will confine ourselves to the case of cylindrical bending.

The simplest versions of the formulations of these problems can be written in the form of the following boundary-value problem, which is already given in the normalized form

$$w_{tt} + g_1 w_t + g_2 w_{txxxx} + w_{xxxx} + cw_x = F(w_t, w_x, w_{tx}, w_{xx}), \quad t \geq 0, \quad x \in [0, 1] \tag{1.1}$$

$$w(t, 0) = w_{xx}(t, 0) = w(t, 1) = w_{xx}(t, 1) = 0 \tag{1.2}$$

Here, $w = w(t, x)$ is the normalized transverse displacement of the middle surface. The constant $g_1 > 0$ is a generalized and normalized damping coefficient and the non-negative coefficient g_2 characterizes the viscoelastic friction. The principal parameter in this problem $c \geq 0$. This parameter is usually proportional to the square of the velocity of the flow and this will be so if the well-known Ackeret formula is used. If aerodynamic forces are taken into account on the basis of the law of plane sections (see Ref. 1, § 4.8), then the parameter c is found to be proportional to the flow velocity. It has been noted that, at high supersonic velocities, both versions are equally adimissible but the version which takes account of aerodynamic forces using. Il'yushin's law of plane sections is more popular in problems of the theory of aero-elasticity. It is also pertinent to point out that the version when $c = \rho$, $g_1 = \sqrt{\delta\rho}$ is adimissible. The flow velocity can be expressed in terms of the dynamic pressure ρ. The coefficient δ is usually fairly small. However, the choice of the actual form for the determination of c is not of fundamental importance in the mathematical analysis.

The right-hand side of Eq. (1.1) reflects the effect of non-linearities on the dynamics of the oscillations and takes account of the non-linear components of the geometric and aerodynamic loads. We shall further assume that F is a fairly smooth function (functional) of its variables while the order of smallness of F is greater than the first order at the origin. So, it is assumed[2] that

$$F(w_t, w_x, w_{tx}, w_{xx}) = \left(b_1 \int_0^1 w_z^2 dz + b_2 \int_0^1 w_z w_{zt} dz\right) w_{xx}$$

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where b_1 and b_2 are non-negative constants characterizing the measure of the non-linear axial (membrane) forces and the viscoelastic friction respectively. Aerodynamic non-linearities are not taken into account in this version. Another version[1] has been proposed

$$F(w_t, w_x, w_{xx}) = b_1\left(\int_0^1 w_z^2 dz\right)w_{xx} - b_3(w_t/c_\infty + Mw_x)^2 - b_4(w_t/c_\infty + Mw_x)^3$$

Here $b_3, b_4 \geq 0$, c_∞ is the speed of sound in the unperturbed medium and M is the Mach number. Aerodynamic forces are taken into account up to cubic terms inclusive on the basis of the law of plane sections ("piston" theory).[1,3] The constants b_3, b_4 depend on the pressure in the unperturbed gas, the polytropic index and the Mach number. A detailed discussion of these questions and of renormalizations can be found, for example, in Ref. 1, § 4.8, 4.12. Equation (1.1) was considered together with boundary conditions (1.2), that is, the conditions for a hinged support. These boundary conditions can be replaced by rigid clamping conditions. Below, we will confine ourselves to the case when there is a bilateral flow with different velocities around the plate. In this case, $b_3 = 0$ (see Ref. 1, Ch. 4). We shall discuss the treatment of cases where there is no quadratic non-linearity.

If the damping coefficient g_1 is a quantity of the order of unity, the mathematical apparatus of the investigation can be separated into two parts. In the linear formulation the critical value $c = c_*$, the flutter rate, is a quantity determined by the following conditions. When $c < c_*$, the zero equilibrium state is stable and, on exceeding this threshold value, it loses stability. Finally, when $c = c_*$, a pair of simple and pure imaginary eigenvalues (EVs) belongs to the stability spectrum and the remaining EVs lie in the half-plane of the complex plane, which is separated by the inequality $\mathrm{Re}\lambda \leq -\gamma < 0$. The positive constant γ depends on g_1.

The non-linear analysis when $c \approx c_*$ is based on the use of the classical Andronov-Hopf theorem which enable us to study bifurcations in the neighbourhood of the zero equilibrium state[4–6] (also, see Ref. 2).

Another problem arises if the coefficients g_1 and g_2 are small. We consider the linear differential operator

$$L(c)\upsilon \equiv \upsilon^{IV} + c\upsilon'; \quad \upsilon(0) = \upsilon''(0) = \upsilon(1) = \upsilon''(1) = 0 \tag{1.3}$$

When $c = 0$, all of the EVs of the linear operator $L(c)$ are simple and positive $\lambda_n(0) = (\pi n)^4$, $n \in N$. The smallest positive value of c for which a multiple eigenvalue first appears in the case of the operator $L(c)$ is denoted by c_0. For small g_1, g_2, it is obvious that $c_0 \approx c_*$ but $c_0 < c_*$ always. So, if $g_2 = 0$ and $c = c_*$, then the EV $\pm i\sigma(\sigma > 0)$ belongs to the stability spectrum, and this means that the linear operator $L(c_*)$ must have an EV $\sigma 2 \mp i\, g_1\sigma$ lying in the "parabola of stability". Hence, as c increases in the case of the operator $L(c)$, multiple EVs first appear and it is only afterwards that complex EVs appear. According to the terminology introduced by Movchan (see Ref. 1, § 4.9), c_0 is the lower critical flutter velocity. In the non-linear formulation when $c \approx c_0$, we arrive at a bifurcation problem when a multiple pair of pure imaginary EVs can belong to the stability spectrum.

Note the lower flutter velocity will be determined without using Bubnov's method, which is conventionally used to investigate of panel flutter both in a linear analysis and when investigating the non-linear problem (for example, see Ref. 1 and 2).

2. Determination of the lower critical flutter velocity

We consider the differential operator $L(c)$ $(c \geq 0)$, that is, operator (1.3) defined for sufficiently smooth functions satisfying the above-mentioned boundary conditions. The set of all $c \geq 0$, for which this operator has real EVs, is denoted by $J(c)$. It is clear that $c = 0$ is included in this set and, therefore, $J(c) \neq 0$.

Lemma 1. Suppose $c \in J(c)$. Then, the inequality $\lambda \geq \pi^4$ is satisfied for any EV λ of the operator L(c).

Proof. The equality

$$\int_0^1 \upsilon''^2 dx = \lambda \int_0^1 \upsilon^2 dx$$

is integrated by parts. Expanding the function $\upsilon(x)$ in a Fourier series $\upsilon(x) = \sum_{n=1}^{\infty} \upsilon_n \sin \pi n x$, we obtain the inequality

$$\int_0^1 (\upsilon'')^2 dx = \frac{\pi^4}{2}\sum_{n=1}^{\infty} n^4 \upsilon_n^2 \geq \frac{\pi^4}{2}\sum_{n=1}^{\infty} \upsilon_n^2 = \pi^4 \int_0^1 \upsilon^2 dx$$

from which the correctness of the lemma follows.

We now consider the differential equation

$$\upsilon^{IV} + c\upsilon' - \lambda\upsilon = 0, \quad \lambda \in J(c) \tag{2.1}$$

Suppose $\mu_{1,2} = \alpha \pm i\beta$, $\mu_{3,4} = -\alpha \pm i\sigma(\sigma = \sqrt{\beta^2 - 2\alpha^2})$ are the roots of the corresponding characteristic equation. Then,

$$c = 4\alpha(\beta^2 - \alpha^2), \quad \lambda = (\alpha^2 + \beta^2)(\beta^2 - 3\alpha^2) \tag{2.2}$$

from which it can be concluded that $\beta^2 > 3\alpha^2$, $\alpha > 0$. The above structure of the roots follows from Vieta's theorem. Using the form of the general solution of Eq. (2.1)

$$\upsilon(x) = \sum_{j=1}^{4} A_j \exp(\mu_j x)$$

and the fact that the function $\upsilon(x)$ satisfies the boundary conditions for a hinged support, we obtain the following system of equations for determining of A_j

$$A_1 + A_2 + A_3 + A_4 = 0, \quad \mu_1^2 A_1 + \mu_2^2 A_2 + \mu_3^2 A_3 + \mu_4^2 A_4 = 0$$

$$q_1 A_1 + q_2 A_2 + q_3 A_3 + q_4 A_4 = 0, \quad \mu_1^2 q_1 A_1 + \mu_2^2 q_2 A_2 + \mu_3^2 q_3 A_3 + \mu_4^2 q_4 A_4 = 0$$

where $q_j = \exp\mu_j$ (j=1, 2, 3, 4). This system has a non-zero solution if its determinant is equal to zero. After some reduction, we obtain the characteristic equation

$$P(\alpha, \beta) = (3\alpha^4 + \beta^2\sigma^2)\,\mathrm{sh}\,\sigma \sin\beta + 2\alpha^2\beta\sigma(\mathrm{ch}\,2\alpha - \cos\beta\,\mathrm{ch}\,\sigma) = 0 \tag{2.3}$$

relating α and β. Constructions in the papers by Movchan (see Ref. 1) have been used in deriving of Eq. (2.3). We now add to them.

Finding the lower critical flutter velocity can be interpreted as a conditional extremum problem: finding the minimum of the function

$$c = Q(\alpha, \beta) \equiv 4\alpha(\beta^2 - \alpha^2)$$

subject to the condition of the constraint (2.3). After this, the necessary conditions for an extremum lead to the equality

$$\frac{\partial P}{\partial \alpha}\frac{\partial Q}{\partial \beta} - \frac{\partial P}{\partial \beta}\frac{\partial Q}{\partial \alpha} = 0 \tag{2.4}$$

Equation (2.4) makes up (2.3) to a system of equations for determining the pair (α, β) for which there can be multiple EVs in the case of the operator $L(c)$. Moreover, equality (2.4) is a necessary condition for the existence of multiple EVs in the case of the operator $L(c)$ (see Ref. 7). A check of the sufficiency will be made below by analysing the corresponding boundary-value problem but after the system of transcendental equations (2.3), (2.4) has been investigated.

The solutions of this system obtained using Seidel's method are presented below. We denote its solutions by α_n, β_n and the values of c and the EV λ of the operator $L(c)$ corresponding to them by c_n, λ_n. It was found that

$$\alpha_1 = 2.4658, \quad \beta_1 = 6.3876, \quad c_1 = 343.36, \quad \lambda_1 = 1051.8$$

We will now show five more sets of these solutions

$$\alpha_2 = 4.9981, \quad \beta_2 = 12.702, \quad c_2 = 2734.3, \quad \lambda_2 = 16096$$

$$\alpha_3 = 7.5564, \quad \beta_3 = 18.867, \quad c_3 = 9187.4, \quad \lambda_3 = 79127$$

$$\alpha_4 = 10.115, \quad \beta_4 = 25.294, \quad c_4 = 21741, \quad \lambda_4 = 2.4542 \cdot 10^5$$

$$\alpha_5 = 12.675, \quad \beta_5 = 31.584, \quad c_5 = 42417, \quad \lambda_5 = 5.9604 \cdot 10^5$$

$$\alpha_6 = 15.241, \quad \beta_6 = 37.853, \quad c_6 = 73240, \quad \lambda_6 = 12.276 \cdot 10^5$$

It can be seen that the approximate equalities $c_n \approx n^3 c_1$, $\lambda_n \approx n^4\lambda_1$ hold with a fairly reasonable degree of relative error (it does not exceed 5% in the majority of cases). However, it can be shown from similarity considerations that the exact equalities $c_n \approx n^3 c_1$, $\lambda_n = n^4\lambda_1$ hold for all $n \in N$.

We will now prove them. Consider the auxiliary boundary-value problem

$$\upsilon^{IV} + c_l\upsilon' = \lambda_l\upsilon, \quad \upsilon = \upsilon(z), \quad z \in [0, l], \quad \upsilon(0) = \upsilon(l) = \upsilon''(0) = \upsilon''(l) = 0$$

For this boundary-value problem, we find those values of c_1 for which it has multiple EVs. The replacement $z = lx$ reduces it to a boundary-value problem in the interval [0,1], where $c = l^3 c_l$, $\lambda = l^4\lambda_l$. Consequently, having determined those c for which there are multiple EVs in the case of the operator $L(c)$, it is possible to find the corresponding pairs (c_l, λ_l). Now, putting, $l = 1/n$ $(n \in N)$, we note that the solution of the auxiliary problem in the interval $[0, 1/n]$ can be extended to the solution of the corresponding boundary-value problem in the interval [0,1]. This last fact is verified, for example, by means of re-expansions in sine series.

It follows from the physical meaning of the formulation of the problem that the smallest values of the free stream velocity are of interest. Hence,

$$c_0 = c_1, \quad \lambda_0 = \lambda_1$$

Standard calculations enable us to find A_1, A_2, A_3, A_4 and, consequently, the characteristic element $e_0(x)$ of the operator $L_0 = L(c_0)$ corresponding to $\lambda = \lambda_0$. It is found that $0.63 \mp 1.07i$, $A_3 = 0.26$, $A_4 = 1$. Moreover, it can be verified by simple integration that the inhomogeneous differential equation

$$h^{IV} + c_0 h' - \lambda_0 h = e_0(x)$$

has a solution $h_0(x)$, which satisfies the boundary conditions for a hinged support. So, the characteristic function $e_0(x)$ and the associated function $h_0(x)$ correspond to the eigenvalue $\lambda = \lambda_0$ and

$$h_0(x) \times 10^4 = \sum_{j=1}^{4} B_j \exp(\mu_j x) + x \sum_{j=1}^{4} C_j \exp(\mu_j x)$$

$$B_{1,2} = 13 \mp 45i, \quad B_3 = 25, \quad B_4 = 0, \quad C_{1,2} = -1 \pm 12i, \quad C_3 = 3, \quad C_4 = 23$$

The differential operator L_0 has a double EV λ_0. It follows from will known results see, for example, Ref. 7, Ch. 1), that its remaining EVs λ_k (k = 1, 2, ...) are simple and the eigenfunctions $e_k(x)$, the explicit form of which is of no importance in the subsequent constructions, correspond to them. The differential operator, which is adjoint to L_0[7] has the same EVs λ_0, λ_1, λ_2, ... and λ_0 is a double EV and the eigenfunction $p_0(x) = e_0(1 - x)$ and the associated function $q_0(x) = e_0(1 - x)$ correspond to it. As previously, the remaining EVs λ_k are single and the eigenfunctions $p_k(x) = e_k(1 - x)$ (k = 1, 2, ...) correspond to them. The hinged support boundary conditions are regular and the system of functions $e_0(x)$, $h_0(x)$, $e_1(x)$, $e_2(x)$, ... therefore forms a Riesz–Bari basis (see Ref. 7). In particular, the equalities

$$(e_0, p_0) = 0, \quad (e_0, q_0) \neq 0, \quad (h_0, p_0) \neq 0, \quad (h_0, q_0) = 0$$

$$(e_0, p_j) = 0, \quad (h_0, p_j) = 0, \quad (e_k, p_j) = 0, \quad \text{если} \quad k \neq j$$

hold. Here, (f, g) is a scalar product in $L_0(0, 1)$.

It is useful to note the following fact for the constructions in the next section. The inequalities $\lambda_k/\lambda_0 \neq 4$, $\lambda_k/\lambda_0 \neq 9$ hold for all $k \in N$. This is verified numerically in the case of small k. From equalities (2.2), we express α and β in terms of λ and c. We obtain

$$\alpha = \sqrt{Q^{1/3} - \frac{\lambda}{12} Q^{-1/3}}, \quad \beta = \sqrt{\alpha^2 + \sqrt{4\alpha^4 + \lambda}}, \quad Q = \frac{c^2}{128} + \sqrt{\left(\frac{\lambda}{12}\right)^3 + \left(\frac{c^2}{128}\right)^2}$$

We choose the real root, after which, initially putting $c_* = c_0$, $\lambda_* = 4\lambda_0$, we substitute the values α_*, β_* which have been found into Eq. (2.3). Checking shows these values of α_*, β_* do not satisfy it. It is also possible to act in a similar manner for $c_* = c_0$.

For sufficiently large k, this can also be verified asymptotically: $\lambda_k \approx (\pi k)^4$. In this case, the same approach of considering the boundary-value problems in the interval $[0, 1/k]$ as was adopted above can be used.

We now consider the following boundary-value problem

$$u_{tt} + u_{xxxx} + c_0 u_x = \exp(i\sigma t)\varphi(x), \quad \sigma \in R \tag{2.5}$$

$$u(t, 0) = u(t, 1) = u_{xx}(t, 0) = u_{xx}(t, 1) = 0 \tag{2.6}$$

where $\varphi(x)$ is a fairly smooth function.

It follows from the preceding constructions that the corresponding homogeneous boundary-value problem has a denumerable number of periodic solutions

$$E_m(t, x) = \exp(i\sigma_m t) e_m(x), \quad (\sigma_m = \sqrt{\lambda_m})$$

and the following assertion also holds.

Lemma 2. When $\sigma \neq \sigma_k$, inhomogeneous boundary-value problem (2.5), (2.6) has a unique periodic solution with period $2\pi/\sigma$. If $\sigma = \sigma_m$ (m = 0, 1, ...), then boundary-value problem (2.5), (2.6) has just one periodic solution if the following equality (solvability condition) holds

$$(\varphi, p_m) = \int_0^1 \varphi(x) p_m(x) dx = 0$$

When $\sigma = \sigma_0 (\sigma_0 = \sqrt{\lambda_0})$ and $\varphi(x) = e_0(x)$, boundary-value problem (2.5), (2.6) has a periodic solution $u(t, x) = \exp(i\sigma_0 t) h_0(x)$.

3. The normal form for the non-linear problem

We will now consider boundary-value problem (1.1), (1.2), assuming that the coefficients g_1 and g_2 are small. It is considered to be permissible and advisable to carry out the following normalization

$$g_1 = 2g_0\varepsilon, \quad g_2 = b_0\varepsilon^2 \tag{3.1}$$

where ε is a small non-negative parameter, and g_0 and b_0 are non-negative constants. We also put

$$c = c_0 + a_0\varepsilon^2, \quad a_0 \in R \tag{3.2}$$

In choosing the parameters of the problem using equalities (3.1) and (3.2), it can be observed that, when $a_0 \leq 0$, $\varepsilon > 0$, $g_0 + b_0 > 0$, the zero solution of boundary-value problem (1.1), (1.2) is already known to be asymptotically stable and, when $a_0 > 0$ ($\varepsilon \neq 0$), it can become unstable if $a_0 > K(g_0, b_0) > 0$. A more detailed analysis of the neighbourhood of the zero equilibrium state in the norm of the phase space of the solutions of boundary-value problem (1.1), (1.2) will be proposed below using the method of normal forms. It is natural to select the following functional spaces: $\overset{\circ}{W}{}_2^4 \times \overset{\circ}{W}{}_2^4$ if $g_2 \neq 0$ and $\overset{\circ}{W}{}_2^4 \times \overset{\circ}{W}{}_2^2$ if $g_2 = 0$ ($b_0 = 0$) as the phase space of the boundary-value problem considered. These problems have been considered earlier[5,6]. Here, the closure of the set of functions from $C^k(0, 1)$ which satisfies the hinged support boundary conditions with respect to the norm of the Sobolev space $W_2^k(0, 1)$ is denoted by $\overset{\circ}{W}{}_2^k \times \overset{\circ}{W}{}_2^k(0, 1)$.

We shall seek solutions of boundary-value problem (1.1), (1.2) in which the coefficients are selected according to the normalizations (3.1), (3.2) in the form

$$w(t, x, \varepsilon) = \varepsilon w_1(t, s, x) + \varepsilon^2 w_2(t, s, x) + \varepsilon^3 w_3(t, s, x) + \varepsilon^4 w_4(t, s, x, \varepsilon) \tag{3.3}$$

where w_j are fairly smooth functions which satisfy boundary conditions (1.2) and $s = \varepsilon t$. We put

$$w_1(t, s, x) = (z_0(s)\exp(i\sigma_0 t) + \bar{z}_0(s)\exp(-i\sigma_0 t))e_0(x)$$

$$w_2(t, s, x) = (\upsilon_0(s)\exp(i\sigma_0 t) + \bar{\upsilon}_0(s)\exp(-i\sigma_0 t))h_0(x) + \sum_{k=1}^{\infty}((z_k(s)\exp(i\sigma_k t) + \bar{z}_k(s)\exp(-i\sigma_k t))e_k(x))$$

The functions $z_k(s)$ ($k \in N$) are such that $\sum_{k=1}^{\infty}|z_k(s)^2 k^8 < \infty$. This inequality guarantees the inclusion of the function $w_2(t, s, x)$, for fixed t and s (that is, as a function of x) in the phase space of the boundary-value problem investigated.

We substitute expression (3.3) into boundary-value problem (1.1), (1.2), in which the normalizations (3.1) and (3.2) have been taken into account, and expand the right-hand and left-hand sides of Eq. (1.1) in powers of ε. As a result, we obtain a recurrent sequence of linear boundary-value problems. Thus, separating the term accompanying ε, we obtain the boundary-value problem

$$w_{1tt} + w_{1xxxx} + c_0 w_{1x} = 0 \tag{3.4}$$

$$w_1(t, s, 0) = w_1(t, s, 1) = w_{1xx}(t, s, 0) = w_{1xx}(t, s, 1) = 0 \tag{3.5}$$

Equalities (3.4) and (3.5) are obviously satisfied in the case of the function $w_2(t, s, x)$. Equating the terms in ε^2, we obtain an inhomogeneous boundary-value problem which differs from problem (3.4), (3.5), with a right-hand side of (3.4) equal to $-2w_{1st} - 2g_0 w_{1r}$.

The above-mentioned function $w_2(t, s, x)$ satisfies this inhomogeneous boundary-value problem if

$$\upsilon_0(s) = -2i\sigma_0(z_0'(s) + g_0 z_0(s)) \tag{3.6}$$

Differentiation with respect to the auxiliary variable s is denoted by a prime.

Finally, for $w_3(t, s, x)$, we obtain the inhomogeneous boundary-value problem with right-hand side

$$G(t, s, x) = F(w_1) - H(t, s, x)$$

$$\begin{aligned} H(t, s, x) = {} & [z_0''(s)q_0(t) + \bar{z}_0''(s)\bar{q}_0(t))e_0(x) + 2i\sigma_0[\upsilon_0'(s)q_0(t) - \upsilon_0'(s)\bar{q}_0(t)]h_0(x) + \\ & + 2i\sigma_0 g_0[\upsilon_0(s)q_0(t) - \bar{\upsilon}_0(s)\bar{q}_0(t)]h_0(x) + 2i\sigma_0 g_0[z_0'(s)q_0(t) - \bar{z}_0'(s)\bar{q}_0(t)]e_0(x) + \\ & + i\sigma_0 b_0(z_0(s)q_0(t) - \bar{z}_0(s)\bar{q}_0(t))e_0^{IV}(x) + a_0(z_0(s)q_0(t) + \bar{z}_0(s)\bar{q}_0(t))e_0'(x) + \\ & + 2g_0\sum_{k=1}^{\infty} i\sigma_k(z_k(s)q_m(t) - \bar{z}_k(s)\bar{q}_m(t))e_k(x), \quad q_m(t) = \exp(i\sigma_m t), \quad m = 0, 1, 2, \dots \end{aligned}$$

Taking account of the fact that $F(w_1) = F(w_{1t}, w_{1x}, w_{1tx}, w_{1xx})$ is a homogeneous cubic form, we refine the structure of this term:

$$\begin{aligned} F(w_1) = {} & F_1 z_0^3(s)\exp(3i\sigma_0) + F_2 z_0^2(s)\bar{z}_0(s)\exp(i\sigma_0 t) + \\ & + F_3\bar{z}_0^2(s)z_0(s)\exp(-i\sigma_0 t) + F_4\bar{z}_0^3(s)\exp(-3i\sigma_0 t) \end{aligned}$$

Here, $e_0(x)$ are known functions of the variable x and their form depends on F_1, F_2, F_3, F_4. It follows from the conditions for the solvability of the inhomogeneous boundary-value problem for $w_3(t, s, x)$ in the class of trigonometric polynomials in the variable t and Lemma 2 that the following assertion holds.

Lemma 3. The functions $z_k(s)$ ($k \in N$) necessarily satisfy the sequence of ordinary differential equations

$$z_k' = -2g_0 z_k \tag{3.7}$$

The equality

$$2i\sigma_0\upsilon_0' I_0 + 2i\sigma_0 g_0\upsilon_0 I_0 + (d_1 + id_2)z_0 - (d_3 + id_4)z_0|z_0|^2 = 0 \tag{3.8}$$

holds for the functions $z_0(s)$ and $\upsilon_0(s)$.

The conditions for the solvability of inhomogeneous boundary-value problem (2.5), (2.6) as well as the inequalities $\sigma_k \neq 3\sigma_0$ ($k \in N$) are used in the deriving of Eqs. (3.7) and (3.8). It follows from Lemma 2 that (integration with respect to x is carried out from 0 to 1 everywhere below)

$$I_0 = \int h_0(x)p_0(x)dx, \quad d_1 + id_2 = a_0I_1 + i\sigma_0 b_0 I_2, \quad d_3 + id_4 = \int F_2(x)p_0(x)dx$$

$$I_1 = \int e_0'(x)p_0(x)dx, \quad I_2 = \int e_0^{IV}(x)p_0(x)dx$$

It is pertinent to recall that the third equality can be specified depending on the choice of non-linearity. For instance, when the non-linearity is chosen in accordance with the known approach,[2] it was found that

$$d_3 + id_4 = (b_1 + i\sigma_0 b_2)I_3I_4, \quad I_3 = \int (e_0'(x))^2 dx, \quad I_4 = \int e_0''(x)p_0(x)dx$$

In the other version of choosing the non-linearity when $b_3 = 0$,[1] we have

$$d_3 + id_4 = b_1I_3I_4 - b_4M^3I_5, \quad I_5 = \int (e_0'(x))^3 p_0(x)dx$$

In all the cases mentioned, the integrals in the formulae for the determining of the coefficients of Eq. (3.8) can be calculated with an accuracy which depends on the accuracy of the calculations for the determining of the functions $e_0(x)$ and $h_0(x)$ (see Section 1). Thus, it is found that

$$I_0 \approx -0.06912, \quad I_1 \approx 218.5, \quad I_2 \approx -75031, \quad I_3 \approx 3736, \quad I_4 \approx 1784, \quad I_5 \approx 8.629 \cdot 10^5$$

In the first place, the sign of these integrals is important for the subsequent constructions. Using equalities (3.6) and (3.8), the equation

$$z_0'' + 2g_0z_0' + (q_1 + iq_2)z_0 + (Q_1 + iQ_2)z_0|z_0|^2 = 0 \tag{3.9}$$

can be derived which determines the dynamics of the solutions of the boundary-value problem in a certain neighbourhood of its zero equilibrium state. Here,

$$q_1 = g_0^2 - a_0I_1r_0, \quad q_2 = b_0\sigma_0I_2r_0, \quad Q_1 = d_1r_0, \quad Q_2 = d_4r_0, \quad r_0 = (4\sigma_0^2|I_0|)^{-1}$$

We recall that the constants d_1 and d_2 are calculated depending on the choice of the non-linearity in the initial boundary-value problem and the formulae for calculating them have been presented earlier. In the first version of choosing the non-linearity considered above, when only geometric non-linearity is taken into account, it can be seen that $Q_1 > 0$ and the interaction of the geometric and aerodynamic non-linearities leads to the fact that the inequality $Q_1 \leq 0$ can be satisfied for Q_1.

Equation (3.9) has been derived earlier[8,9] at a phenomenological level using, in particular, Bubnov's method. In this paper, it has been obtained as the normal form of boundary-value problem (1.1), (1.2).

4. Analysis of the normal form

We will now consider the question of the existence and stability of periodic solutions of Eq. (3.9) of the form

$$z_0(s) = \sqrt{\eta}\exp(i\omega s) \tag{4.1}$$

where, naturally, $\eta > 0$. After substituting expression (4.1) into Eq. (3.9) and separating the real and imaginary parts for η and ω, we obtain the system of equations

$$-\omega^2 + Q_1\eta + q_1 = 0, \quad 2g_0\omega + Q_2\eta + q_2 = 0 \tag{4.2}$$

Expressing ω from the second equation of system (4.2), we obtain the equation for the non-negative amplitude of η

$$P(\eta) = Q_2^2\eta^2 + (2q_2Q_2 - 4g_0^2Q_1)\eta + (q_2^2 - 4g_0^2q_1) = 0 \tag{4.3}$$

In the special case when $Q_2 = 0$ and $q_2 = 0$, it is not a question of the periodic solution of (4.1) but of a non-zero equilibrium state of Eq. (3.9).

Suppose η_* is a simple positive root of Eq. (4.3). We will consider the question of stability of periodic solution (4.1) corresponding to it. For this purpose, we put

$$z_0(s) = \sqrt{\eta_*}\exp(i\omega_*s)(1 + u(s)), \quad u(s) = u_1(s) + iu_2(s), \quad \omega_* = -(Q_2\eta_* + q_2)/(2g_0)$$

After substitution into Eq. (3.9), linearization and separation of the real and imaginary parts, we obtain a system of two linear differential equations for the now real functions $u_1(s)$ and $u_2(s)$

$$u_1'' + 2(g_0u_1' - \omega_*u_2') + 2Q_1\eta_*u_1 = 0, \quad u_2'' + 2(g_0u_2' + \omega_*u_1') + 2Q_2\eta_*u_1 = 0 \tag{4.4}$$

for which it is necessary to investigate the stability of the zeroth equilibrium state. This question can be reduced in the usual way to the investigation of the characteristic equation

$$\lambda^4 + p_3\lambda^3 + p_2\lambda^2 + p_1\lambda = 0 \tag{4.5}$$

where

$$p_1 = 4(g_0Q_1\eta_* + Q_2\eta_*\omega_*), \quad p_2 = 4g_0^2 + 2Q_1\eta_* + 4\omega_*^2, \quad p_3 = 4g_0$$

It is understandable that one of the roots of the characteristic equation is equal to zero. This reflects the fact that Eq. (3.9) has a periodic solution. The arrangement of the three remaining roots of Eq. (4.5) determines the stability of the cycle. Simple calculations show that

$$p_1 = -\eta_* g_0^{-1} P'(\eta)|_{\eta = \eta_*} \tag{4.6}$$

First suppose $Q_2 \neq 0$.

Lemma 4. Suppose Eq. (4.3) has two positive roots η_1 and η_2. Then, the periodic solution corresponding to the greater root is necessarily unstable. If Eq. (4.3) has only one positive root, then the periodic solution of differential equation (3.9) corresponding to it is unstable.

The correctness of this assertion follows from the obvious fact that the inequality $P'(\eta)|_{\eta=\eta_2} > 0$ holds in the case of the larger root η_2 of quadratic equation (4.3). It follows from formula (4.6) that, in this case, $p_1 < 0$. A similar assertion has been presented earlier[8,9].

Now, suppose $Q_2 = 0$, which occurs in the case of the second of the versions for choosing the non-linear terms which have been considered. In this case, $q_2 = 0$. The differential equation (3.9) then has a family of non-zero equilibrium states

$$z_0(s) = z_0 = \sqrt{\eta}\exp(ih), \quad \eta = -\eta_1 Q_1^{-1} > 0, \quad h \in R \tag{4.7}$$

Lemma 5. Suppose $q_1 < 0$ ($Q_1 > 0$). Then, any of the equilibrium states of family (4.7) is stable and the zero equilibrium state (3.9) is unstable. When $q_1 > 0$ ($Q_1 < 0$), solutions (4.7) are unstable and the zero equilibrium state is stable.

Thus, here a non-equilibrium state bifurcates from the zero solution of Eq. (3.9) with a change of stability. The bifurcations in the Andronov-Hopf theorem are of this kind.

The proof of Lemma 5 follows from an analysis of Eq. (4.5) as well as from a standard analysis of the stability of the zeroth equilibrium state of Eq. (3.9).

We further assume that $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is a fairly small positive constant and η_* is a simple root of Eq. (4.3).

Theorem. Suppose $z^*(s) = \sqrt{\eta_*}\exp(i\omega_* s)$ is a structurally stable self-similar cycle of differential equation (3.9). A periodic solution of boundary-value problem (1.1), (1.2) for which the asymptotic formula

$$w(t, x, \varepsilon) = \varepsilon[\exp(i\sigma_\varepsilon t) + \exp(-i\sigma_\varepsilon t)]\sqrt{\eta_*}e_0(x) +$$
$$+ \varepsilon^2[(g_0 + i\sigma_0)\exp(i\sigma_\varepsilon t) + (g_0 - i\sigma_0)\exp(-i\sigma_\varepsilon t)]\eta_* h_0(x) + o(\varepsilon^2), \quad \sigma_\varepsilon = \sigma_0 + \varepsilon\omega_*$$

holds can then be set to correspond to each such cycle. The cycle of boundary-value problem (1.1), (1.2), which is generated by this solution, inherits the stability properties of the cycle $z_*(s)$ of Eq. (3.9).

The method proposed here for investigating of the periodic solutions in connection with problem (1.1), (1.2) is in the nature of an adaptation of the well-known Krylov–Bogolyubov method[10] in connection with problems with an infinite-dimensional phase space. Its proof exists for a wide class of such equations.[11–14] Note that an even more general problem of the bifurcation of invariant tori has been considered (see for example, Ref. 11, § 4) but, here, it refers solely to the periodic solutions. In the detailed proof, use is made of the fact that the solutions of system (3.7) tend to zero at an exponential rate.

We put

$$w = u_1, \quad \dot{w} = u_2, \quad u = \mathrm{col}(u_1, u_2)$$

We then obtain the equation for u

$$\dot{u} = B_0 u + \varepsilon B(\varepsilon)u + R(u)$$

$$B_0 = \begin{Vmatrix} 0 & I \\ L_0 & 0 \end{Vmatrix}, \quad B(\varepsilon) = \begin{Vmatrix} 0 & 0 \\ C(\varepsilon) & D(\varepsilon) \end{Vmatrix}, \quad R(u) = \begin{Vmatrix} 0 \\ -F \end{Vmatrix}$$

$$C(\varepsilon)u = -a_0\varepsilon u_{1x}, \quad D(\varepsilon)u = -2g_0u_2 - b_0\varepsilon u_2, \quad F = F(u_2, u_{1x}, u_{2x}, u_{1xx})$$

which occurred in the class of equations considered earlier.[11,12] We note that the linear operator B_0 generates a subgroup of linear bounded operators of the class (C_0),[15] which follows from the correct solvability of problem (3.4), (3.5), supplemented with the initial conditions from the functional spaces indicated above. The operators $C(\varepsilon)$ and $D(\varepsilon)$ are governed by the operator L_0.[15] The equation $\dot{u} = B_0 u$ has a deunmerable number of periodic solutions with frequencies $\sigma_0, \sigma_1, \ldots$.. These solutions have been written out in Section 3, and it has been checked that there are no lower resonances, which is necessary for the results presented earlier to be applicable.[11,12] In the case being considered, it always reduces to checking the inequalities $\sigma_k/\sigma_0 \neq 2, 3$. There is one generating frequency here and it is equal to σ_0.

5. Conclusion

It has been shown that, in the case of small damping coefficients, oscillations can arise at velocities which are lower than the flutter velocity ($c \approx c_0$, $c_0 < c_*$). The bifurcating periodic solutions can be stable and unstable. Instability is more typical. Actually, if Eq. (3.9) has two periodic solutions, the solution which is greater in amplitude is always unstable. When $c < c_*$, the zero solution of boundary-value problem (1.1), (1.2) which has been linearized in the zero equilibrium state, is always unstable but the treatment of the problem in a non-linear formulation shows that there are unstable periodic solutions in a sufficiently small neighbourhood of the equilibrium state. In the case considered, their amplitude is of the order of ε. An increase in the amplitudes of the small initial deflections is therefore possible as time passes, that is, a version of the strong excitation of oscillations is possible. According to the terminology of Andronov and his students, in this problem the boundary of stability in the domain of the parameters is "dangerous".[16]

This conclusion is only reached because of the non-linear formulation of the problem and it remains in force for the different ways of taking non-linear effects into accout. It is also pertinent to note that, after renormalization, the damping coefficients can also turn out to be small for sufficiently large cylindrical stiffness of a plate.

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References

1. Bolotin VV. *Non-conservative Problems of the Theory of Elastic Stability*. Oxford: Pergamon; 1963.
2. Guckenheimer J, Holmes F. *Non linear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. New York: Springer; 1983.
3. Il'yushin AA. The law of plane sections in the aerodynamics of high supersonic velocities. *Prikl Mat Mekh* 1956;**20**(6):733–55.
4. Kulikov AN, Liberman BD. A New approach to the investigation of problems of non-linear panel flutter. *Vestn Yarosl Univ* 1976;**6**:118–39.
5. Kolesov VS, Kolesov YuS, Kulikov AN, Fedik II. A mathematical problem in the theory of elastic stability. *Prikl Mat Mekh* 1978;**42**(3):458–65.
6. Holmes PJ, Marsden JE. Bifurcations to divergence and flutter in flow-induced oscillations: an infinite-dimensional analyais. *Automatica* 1978;**14**(4):367–84.
7. Neumark MA. *Linear Differential Operators*. Berlin: Akademie; 1963.
8. Kulikov AN. Non-linear panel flutter: the danger of strong excitation of oscillations. *Differents Uravneniya* 1992;**28**(6):1080–2.
9. Kulikov AN. An analogue of the Hopf bifurcation theorem in the problem of the mathematical investigation of non-linear panel flutter with a small damping coefficient. *Differents Uravneniya* 1993;**29**(5):780–95.
10. Bogolyubov N, Mitropol'ky Y. *Asymptotic Methods in the Theory of Non-Linear Oscillations*. Delhi: Hindustan; 1961.
11. Kolesov AYu, Kulikov AN. Invariant Tori of Non-Linear Evolutionary Equations. *Yaroslavl:Izd Yaroslav Univ* 2003.
12. Kolesov AYu, Rozov NKh. *Invariant Tori of Non-Linear Wave Equations*. Moscow: Fizmatlit; 2004.
13. Kolesov AYu, Kulikov AN, Rozov NKh. Invariant tori of one class of point mappings: the ring principle. *Differents Uravneniya* 2003;**39**(5):584–601.
14. Kolesov AYu, Kulikov AN, Rozov NKh. Invariant tori of a class of point mappings:conservation of an invariant torus under perturbations. *Differents Uravneniya* 2003;**6**:738–53.
15. Krein SG. *Linear Differential Equations in Banach Space*. Boston: Birkhauser; 1967.
16. Bautin NN, Leontovich EA. *Methods and Procedures for the Qualitative Investigation of Dynamical Systems in a Plane*. Moscow: Nauka; 1990.

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